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LETTER TO THE EDITOR

Squeezing of light within the framework of the population theoretic approach

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Abstract. This letter proposes a population point process model of cavity radiation in which the process of spontaneous emission evolves as a multiphase birth process. The heterogeneity arising from the emissions in different phases lends itself to an interpretation of superposition of photons in different streams. Squeezing is shown to be a natural consequence of the heterophase evolution.

The object of this letter is to establish the squeezing of light within the framework of the population theoretic approach to cavity radiation and detection. The earliest attempt to describe fluctuation in amplification of quanta was by Shimoda *et al* [1] who used the population growth model as the basis for the evolution of photon population; the parameters of the population evolution were then used to interpret physical characteristics like amplification, attenuation and thermal equilibrium. This was followed by Shepherd and Jakeman [2-4] who incorporated effects due to interaction between the resulting radiation (field) and the detector. In particular Shepherd [2] characterised the population process as a Markov process with constant evolution parameters and identified the Gaussian-Lorentzian nature of the resulting thermal field. Recently non-Markov population evolution models have been proposed by Srinivasan *et al* [5-8] to establish the viability of the population theoretic approach in general and to confirm that the theory is able to bring out, on the one hand, diverse features such as the non-Lorentzian nature of the spectrum of the resulting field and, on the other, non-classical features such as antibunching observed in typical resonant fluorescence. More recently Srinivasan and Sridharan [9] have proposed a multiphase evolution of population in which the state process governing spontaneous emission is modelled as a semi-Markov process which in turn makes the resulting field correspond to the photons from a stream, obtained by amplitude mixing of the most general type, of coherent and chaotic beams. In the present letter we proceed further on these lines and establish that squeezing (see for example [10]) can also be accommodated within the framework of population theory. This should not cause any surprise inasmuch as particle (field) characteristics of light, including the statistics, are incorporated in a typical population theory.

It is convenient to use the Shepherd model [2] as adapted by Srinivasan [8] as the starting point; the photon field is modelled as a discrete-valued population process, the field evolving in cavity as a birth (stimulated emission), death (absorption) and immigration (spontaneous emission). The field detector interaction is modelled as an emigration process with a constant rate η per individual (photon). Likewise the death (cavity absorption) rate is assumed to be a constant; however, the emission rates are not constants and are specified as follows. The process of spontaneous emissions, taken in isolation, is governed by a semi-Markov process $\{Z(t)\}$ over a finite set of elements $1, 2, \ldots, m$ with constant rates β_i of transition, non-vanishing only for transition of the type $i \to i+1$ $(i = 1, 2, \ldots, m-1)$, $m \to i$ $(i = 1, 2, \ldots, m)$; the spontaneous emission of a photon occurs when $Z(t)$ makes a transition from state m. We use the index i to characterise the photon thus emitted. Likewise the process of stimulated emission due to any particular photon (of index i) is taken to be a general point process of emissions in which the time to the first emission is the sum of $(n+1)$ positive independent random variables each with a negative exponential distribution with parameter λ_i^j $(j = 1, 2, \ldots, n+1)$, with subsequent emissions occurring at a rate $\alpha_i = \lambda_i^{n+1}$. Thus each of the photons of type i is assumed to evolve in time, independent of each other, through a series of phases, the sojourn through the first n phases being completed before the actual emission, the rate of emission itself being a constant α_i. The emitted photons carry the same index i and in return repeat the process independent of other photons. The (cavity) absorption rate is assumed to be a constant equal to μ for all types i and all the phases except the last where it is taken to be equal to $\lambda_i^n + \mu$; the particular differential choice is made to facilitate thermal equilibrium whenever it is needed. As emphasised elsewhere [6], the evolution through phases is only a simple device to handle non-Markov emission processes which are otherwise intractable. The particular case when $m = 1$ was dealt with earlier [7] and led to a thermal stream of photons. To proceed further, we use t as the time parameter and introduce the following notation:

$X_i(t)$: the size of the population of photons of index i $(i = 1, 2, \ldots, m)$

$X(t)$: the total size of the photon population

$Z(t)$: the state process of spontaneous emission taking values over the index set $1, 2, \ldots, m$

$X_i^j(t)$: the size of the population of photons of index i and phase j $(j = 1, 2, \ldots, n;$ $i = 1, 2, \ldots, m)$

$g_j(w, t)$: conditional generating function of photons defined to be equal to

$$E[w^{X_i(t)} \mid X(0) = X_i^j(0) = 1, \beta_k = 0, k = 1, 2, \ldots, m] \qquad i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n \tag{1}$$

$G_i(w_1, w_2, \ldots, w_m, t)$: generating function of photons of different indices denoted for brevity by $G_i(t)$ where

$$G_i(t) = E[w_1^{X_1(t)} w_2^{x_2(t)} \ldots w_m^{X_m(t)} \mid X(0) = 0, z(0) = i] \qquad i = 1, 2, \ldots, m \tag{2}$$

where E stands for the mathematical expectation of the quantity within the brackets. The special conditioning introduced for $g_j(w, t)$ renders the contributing process immigration taboo and facilitates the solution (see for example [11]); it is also to be noted that $g_j(w, t)$ is independent of i since the population evolution parameters for the sub-process are independent of i. We note that the exponential nature of the distribution of the lifespan of the phases leads to

$$\frac{\partial g_j(w, t)}{\partial t} = -(\lambda_j + \mu + \eta) g_j(w, t) + \lambda_j g_{j+1}(w, t) + \mu + \eta \qquad j = 1, 2, \ldots, \eta - 1 \tag{3}$$

$$\frac{\partial g_n(w, t)}{\partial t} = -(\lambda_n + \mu + \eta + \alpha) g_n(w, t) + \alpha g_n(w, t) g_1(w, t) + \mu + \eta + \lambda_n \tag{4}$$

with the initial condition

$$g_i(w,0)=w \qquad j=1,2,\ldots,n. \tag{5}$$

On the other hand if we examine the process $\{Z(t)\}$ we find that the generating functions $G_i(t)$ satisfy the following set of equations:

$$\frac{\partial G_i(t)}{\partial t}=-\beta_iG_i(t)+\beta_iG_{i+1}(t) \qquad i=1,2,\ldots,m-1 \tag{6}$$

$$\frac{\partial G_m(t)}{\partial t}=-\beta_mG_m(t)+\sum_{i=1}^{m}\nu_iG_i(t)g_1(w_i,t) \tag{7}$$

with the initial conditions

$$G_i(0)=1 \qquad i=1,2,\ldots,m \tag{8}$$

where β_m is given by

$$\beta_m=\nu_1+\nu_2+\ldots+\nu_m. \tag{9}$$

Although the above set of equations is not capable of being solved explicitly, the moment structure can be readily deduced. In fact the moment structure of equations (3) and (4) subject to initial condition (5) is readily available in [7].

At the outset we note that all that we need for our discussion is contained in the first two moments of the steady state photon population. Introducing the notation

$$\begin{aligned} A_{ij}(t)&=\frac{\partial G_i(t)}{\partial w_j} & B_{ijk}(t)&=\frac{\partial^2 G_i(t)}{\partial w_j\,\partial w_k}\\ a_j(t)&=\frac{\partial g_j(w,t)}{\partial w} & b_j(t)&=\frac{\partial^2 g_j(w,t)}{\partial w^2} \end{aligned} \tag{10}$$

where the derivatives on the RHS are evaluated at the point $w=w_1=w_2=\ldots=w_n=1$, we find from equations (3), (4) and (7) that the moments satisfy a set of linear equations in the highest order and hence can be readily solved by the Laplace transform technique. We skip the details and give the final solution† for certain special cases.

(i) n arbitrary: $\lambda_i=\lambda$ $(i=1,2,\ldots,n)$.

$$\begin{aligned} &a_j(t)=e^{-(\mu+\eta)t} & &j=1,2,\ldots,n\\ &A^*_{ij}(s)=A_i(s)\nu_j/\beta_m & &b^*_1(s)=2\lambda^n/\{(s+2\mu+2\eta)[(s+\mu+\eta)^n-\lambda^n]\}. \end{aligned} \tag{11}$$

(ii) $m=2$.

$$\begin{aligned} &A_1(s)=\beta_1\beta_2/s(s+\mu+\eta)(s+\beta_1+\nu_1)\\ &A_2(s)=A_1(s)(s+\beta_1)/\beta_1 \qquad B^*_{2ij}(s)=B^*_{1ij}(s)(s+\beta_2)/\beta_2 \end{aligned} \tag{12}$$

$$B^*_{1ij}(s)=\beta_1\{\nu_i\nu_j[A_j(\mu+\eta)+A_i(\mu+\eta)]+\delta_{ij}\beta_2\nu_ib^*_1(s)\}/s\beta_2(s+\beta_1+\nu_1) \tag{13}$$

$$B_{ii}=B_{kii}(\infty)=\nu_i\beta_1[b^*_1(0)+2\nu_iA_j(\mu+\eta)/\beta_2]/(\beta_1+\nu_1) \tag{14}$$

$$B_{12}=B_{21}=B_{k21}(\infty)=B_{k12}(\infty)=\beta_1\nu_1\nu_2[A_1(\mu+\eta)+A_2(\mu+\eta)]/\beta_2(\beta_1+\nu_1). \tag{15}$$

† We use * as a superscript to denote the Laplace transform.

(iii) $m=3$.

$$A_1(s)=\beta_1\beta_2\beta_3/[s(s+\mu+\eta)D(s)]$$

$$A_2(s)=A_1(s)(s+\beta_1)/\beta_1 \qquad A_3(s)=A_2(s)(s+\beta_2)/\beta_2$$

$$D(s)=s^2+s(\nu_1+\beta_1+\beta_2)+\nu_1(\beta_1+\beta_2)+\beta_1\beta_2 \tag{16}$$

$$B^*_{1ij}(s)=\beta_1\beta_2\nu_i\{\delta_{ij}b^*_1(s)+\nu_j[A_j(\mu+\eta)+A_i(\mu+\eta)]/\beta_3\}/sD(s) \tag{17}$$

$$B_{ii}=B_{kii}(\infty)=\beta_1\beta_2\nu_i\{b^*_1(0)+2\nu_iA_i(\mu+\eta)/\beta_3\}/D(0). \tag{18}$$

Now we are comfortably placed to draw many useful conclusions. Confining our attention to the case $m=2$, we first set $\lambda=0$ so as to render the resulting radiation close to a coherent beam of light. If we further set $\nu_1=0$, the resulting population of photons is indeed coherent (Poisson); on the other hand if we set $\nu_2=0$, the resulting radiation is antibunched with the bunching factor $(\beta_1+\nu_1)/(\mu+\eta+\beta_1+\nu_1)$. Returning to the general case when neither ν_1 nor ν_2 is zero and setting $\nu_1=L\beta_1$, $\mu+\eta=K\beta_1$ and $\nu_2=M\beta_1$, we note that the steady state population of photons forms two streams with

$$\begin{aligned}
&E[X_1(\infty)]=A_{i1}(\infty)=L/K(L+1)\\
&E[X_2(\infty)]=A_{i2}(\infty)=M/K(L+1)\\
&E[X_1(\infty)\{X_1(\infty)-1\}]=B_{11}=L^2/K^2(L+1)(K+L+1)\\
&E[X_2(\infty)\{X_2(\infty)-1\}]=B_{22}=(K+1)M^2/K^2(L+1)(K+L+1)\\
&\mathscr{B}_{11}=B_{11}/[A_{i1}(\infty)]^2=\frac{L+1}{K+L+1}\\
&\mathscr{B}_{22}=B_{22}/[A_{i2}(\infty)]^2=\frac{(L+1)(K+1)}{K+L+1}.
\end{aligned} \tag{19}$$

Hence we infer that the stream corresponding to X_1 is squeezed inasmuch as $\mathscr{B}_{11}<1$ and $\mathscr{B}_{22}>1$ for all possible choices of L, $K>0$. It is interesting to note that the amount of antibunching for stream 1 is independent of M. If we choose $M=1$, it is indeed possible to prove that the total population enjoys a Poisson distribution; thus we conclude that the resulting stream corresponds to coherent light.

To interpret the result, we note that in the population theoretic approach, a light beam is conceived of as an assembly of photons whose evolution is the only characteristic that is available for identification of the diverse properties of the beam. In this connection it is pertinent to note that the phase characteristics of light propagation can be brought out [12] by the third-order correlation of counting statistics; more recently [9] the second-order characteristics of counting statistics of amplitude mixture of chaotic light beams were extracted from the multiphase evolution of the population of photons. To make connection to squeezed light, we note that in the population approach, the quadrature phases make their presence felt through the multiphase evolution which is heterogeneous in the sense that photons of different streams are correlated through the non-Markov evolution of the immigration (spontaneous emission) process; squeezing in the present case makes itself manifest by antibunching in one stream and bunching in another stream. The phenomenon of antibunching was earlier [8] shown to be a consequence of the special type of population evolution; the results presented above go to demonstrate squeezing, although there are limits to the amount of squeezing which actually depends on the number of phases. Since the number of phases can in principle be arbitrary, the amount of squeezing can also be arbitrary. If we choose $K=3/4$, $L=1$, we find $A_{i1}(\infty)=A_{i2}(\infty)=2/3$, $\mathscr{B}_{11}=16/19$ and $\mathscr{B}_{22}=28/19$, establishing antibunching for one of the streams when the mean number is sizeable. Thus we can conclude that a special model of multiphase evolution produces a stream of photons that corresponds to squeezed coherent light.

Returning to the general setting $\lambda\neq 0$, we note that the resulting radiation can be rendered thermal to at least second-order statistics [5] for $\nu_1=0$ and $L=N^2/(K+2n)$. If on the other hand $\nu_2=0$, the resulting radiation still yields bunched statistics. The general case when neither ν_1 nor ν_2 vanishes can be handled with the help of the formulae (16)-(18); the bunching ratios are given by

$$\begin{aligned}\mathscr{B}_{11}&=(L+1)/(K+L+1)+N^2(1+1/L)/(K+2N) \qquad (\lambda=N\beta_1)\\ \mathscr{B}_{22}&=(K+1)(L+1)/(K+L+1)+N^2(1+1/L)/[M(K+2N)].\end{aligned} \tag{20}$$

Since the total population of photons in this case generates a thermal stream, we can conclude that antibunching in one stream and bunching in the residual stream corresponds to the characteristics of thermally squeezed light.

We finally take the case $m=3$; if we choose $\beta_1=\beta_2$, $\nu_1=L\beta_1$, $\nu_2=M\beta_1$, $\nu_3=J\beta_1$, $\lambda=0$, we find

$$\begin{aligned}\mathscr{B}_{11}&=(2L+M+1)/[L+(K+1)(K+L+M+1)]\\ \mathscr{B}_{22}&=\mathscr{B}_{11}(K+1) \qquad \mathscr{B}_{33}=\mathscr{B}_{11}(K+1)^2.\end{aligned} \tag{21}$$

We can maintain a hierarchy [10] by adding the photons of indices 1 and 2 and denoting the bunching ratio by $\mathscr{B}_{12}$, we find

$$\mathscr{B}_{12}=\mathscr{B}_{11}[1+KM/(L+M)]. \tag{22}$$

For $K=L=M=1$, we have $\mathscr{B}_{11}=4/9$, $\mathscr{B}_{12}=2/3$, $\mathscr{B}_{22}=8/9$, $\mathscr{B}_{33}=16/9$, providing ample evidence for bimodal squeezing. It can be verified that the total population is Poisson to second-order statistics provided we choose $J=6/7$. The value of J should not cause any surprise if we make note of the fact that in a two-stream population $(m=2)$, the corresponding parameter had a value 1 and hence a lower value of J is necessary to offset the reduction in the fluctuation of photons of streams 1 and 2. Full details relating to the structure of the various formulae and proof of statements as well as the Poisson nature of the total population for the two-stream case will be presented elsewhere.

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